



BPS equations and the stress tensor

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ABSTRACT

We exploit the relationship between the space components T_{ij} of the energy–momentum tensor and the supercurrent to discuss the connection between the BPS equations and the vanishing of the components of the stress tensor in various supersymmetric theories with solitons. Using the fact that certain combination of supercharges annihilate BPS states, we show that $T_{ij} = 0$ for kinks, vortices and dyons, displaying the connection between supersymmetry and non-interacting BPS solitons.

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1. Introduction

First order BPS equations were originally obtained either by looking for a bound of the soliton mass [1] or by imposing the stress tensor to vanish [2]. Already in this last work the relation between supersymmetry and the possibility of reducing the second order equations of motion to BPS equations at certain critical values of the coupling constants was stressed and afterwards exploited in the search of classical solutions to two-dimensional supersymmetric models [3].

The origin of such a connection was finally clarified by Witten and Olive [4] by considering the supersymmetric extension of bosonic models exhibiting topological soliton solutions. Studying the supersymmetry algebra, it was shown in this work that the soliton topological charge can be identified with the central charge of the supercharge algebra and gives a lower bound for the soliton mass. This was done for the supersymmetric version of a scalar field theory in $1 + 1$ dimensions with kink solutions and a $\mathcal{N} = 2$ Yang–Mills theory in $3 + 1$ dimensions with dyon solutions. Afterwards, the case of vortices in $\mathcal{N} = 2$ supersymmetric gauge theories in $2 + 1$ dimensions and instantons in 4-dimensional Euclidean space was discussed along the same lines [5–7] and the extension to the case of supergravity models was also studied [8]. The question on how supersymmetry protects the Bogomol'nyi bound at the quantum level also deserved a lot of attention [9,10].

We show in the present Letter how the alternative derivation of BPS equations from the vanishing of the soliton stress-tensor T_{ij} ($i, j = 1, 2, 3$) can be also understood supersymmetry point

of view studying the supercurrent-supercharge algebra. As it is well known, the algebra of supersymmetry itself already imposes a very intimate relationship between the supercurrent and the stress tensor. This relationship stems from the connection between the energy–momentum tensor and the supercharge [11]. In fact, both the supercurrent and $T_{\mu\nu}$ must belong to the same supermultiplet and then it is not difficult to understand how an identity between the stress tensor and an appropriate trace containing the supersymmetric transform of the supercurrent connects BPS states and the condition

$$\langle \text{BPS} | T_{ij} | \text{BPS} \rangle = 0. \quad (1)$$

In order to construct the supercurrent and show how its connection with the energy–momentum tensor leads to Eq. (1) we will work with specific models having BPS $(1 + 1)$ -dimensional kinks and $2 + 1$ vortices and also explain how the results can be easily extended to the case to BPS dyons in $3 + 1$ dimensions. In fact, our derivation indicates that the same result should be also valid for any other model with BPS soliton solutions.

It should be mentioned that our work was prompted by a recent work of Manton [12] where new scaling identities for solitons are derived in terms of the stress tensor, showing the relevance of T_{ij} in connection with the study of soliton solutions. As mentioned above, already in the case of vortices it was recognized [2] that the critical point at which the topological bound for the energy of the Abelian Higgs vortices is saturated corresponds to the limiting value between type-I and type-II superconductivity, precisely where forces between vortices (and hence the surface integral of T_{ij}) vanish [13]. We shall see below that supersymmetry provides a way to construct models where general noninteracting solitons equations can be studied by analyzing the Noether supercurrent.

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2. Scalar field theory in two dimensions

The action for the simplest two-dimensional supersymmetric model admitting solitons in its bosonic sector reads, in component fields [14],

$$S = \int d^2x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{i}{2} \bar{\psi} \partial \psi + \frac{1}{2} F^2 + FV[\phi] - \frac{1}{2} V'[\phi] \bar{\psi} \psi \right) \quad (2)$$

with ϕ a real scalar field, ψ a 2-component Majorana spinor, F an auxiliary field and $V[\phi]$ an arbitrary function. We take the metric $g_{\mu\nu}$ with signature $(+, -)$ and the Dirac matrices as

$$\gamma^0 = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

With this conventions the charge conjugation matrix satisfying $C \cdot \gamma^\mu \cdot C^{-1} = -\gamma^\mu$ is given by $C = -\gamma^0$. Given a spinor

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \quad \bar{\psi} = \psi^\dagger \gamma^0$$

the charge conjugate ψ^c is then

$$\psi^c = C \bar{\psi}^T = \psi^*$$

so that ψ_+ and ψ_- are real and

$$\bar{\psi} = i(\psi_-, -\psi_+).$$

The energy-momentum tensor associated with action (2) takes the form

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \frac{i}{2} \bar{\psi} \gamma_\mu \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} (\partial_\alpha \phi \partial^\alpha \phi - V^2 + i \bar{\psi} \gamma^\alpha \partial_\alpha \psi - V' \bar{\psi} \psi) \quad (3)$$

and its symmetric on-shell components

$$T_{00} = \frac{1}{2} ((\partial_0 \phi)^2 + (\partial_1 \phi)^2) + \frac{1}{2} V^2 + \frac{1}{2} V' \bar{\psi} \psi - \frac{i}{2} \bar{\psi} \gamma^1 \partial_1 \psi,$$

$$T_{11} = \frac{1}{2} ((\partial_0 \phi)^2 + (\partial_1 \phi)^2) - \frac{1}{2} V^2 - \frac{1}{2} V' \bar{\psi} \psi + \frac{i}{2} \bar{\psi} \gamma^0 \partial_0 \psi, \quad (4)$$

$$T_{01} = \partial_0 \phi \partial_1 \phi + \frac{1}{2} \psi \gamma^0 \partial_1 \psi, \quad (5)$$

$$T_{10} = \partial_0 \phi \partial_1 \phi - \frac{1}{2} \psi \gamma^1 \partial_0 \psi. \quad (6)$$

The (off-shell) supersymmetric transformations leaving action (2) invariant are

$$\begin{aligned} \delta \phi &= \bar{\epsilon} \psi, \\ \delta \psi &= -i \not{\partial} \phi \epsilon + F \epsilon, \\ \delta F &= -i \bar{\epsilon} \not{\partial} \psi, \end{aligned} \quad (7)$$

and the associated conserved supercurrent is

$$J_\mu = (\not{\partial} \phi + iV) \gamma_\mu \psi, \quad (8)$$

where the auxiliary field has been eliminated using its equation of motion. More explicitly,

$$\begin{aligned} J_0 &= (\not{\partial} \phi + iV) \gamma^0 \psi = \begin{pmatrix} (\partial_- \phi) \psi_+ + V \psi_- \\ (\partial_+ \phi) \psi_- - V \psi_+ \end{pmatrix}, \\ J_1 &= -(\not{\partial} \phi + iV) \gamma^1 \psi = \begin{pmatrix} -(\partial_- \phi) \psi_+ + V \psi_- \\ (\partial_+ \phi) \psi_- + V \psi_+ \end{pmatrix}. \end{aligned} \quad (9)$$

The chiral components Q^\pm of the supersymmetry charge take then the form

$$\begin{aligned} Q_+ &= \int dx \{ (\partial_- \phi) \psi_+ + V \psi_- \}, \\ Q_- &= \int dx \{ (\partial_+ \phi) \psi_- - V \psi_+ \}, \end{aligned} \quad (10)$$

with $\partial_\pm = \partial_0 \pm \partial_1$. Concerning \bar{Q} , one has $\bar{Q} = Q^\dagger \gamma^0 = (iQ_-, -iQ_+)$.

The equal-time commutations/anticommutation relations are

$$\begin{aligned} [\phi(x), \partial_0 \phi(x')] &= i\delta(x - x'), \\ \{\psi_+(x), \psi_+(x')\} &= \delta(x - x'), \\ \{\psi_-(x), \psi_-(x')\} &= \delta(x - x'). \end{aligned} \quad (11)$$

From this, one finds for the supercharge algebra (in the rest frame)

$$\{Q_\pm, Q_\pm\} = 2(M \pm Z), \quad (12)$$

where

$$M = \int dx T_{00}. \quad (13)$$

Concerning Z , it is given by

$$Z = \int dx V[\phi] \frac{\partial \phi}{\partial x} = \int dx \frac{\partial W}{\partial x}, \quad (14)$$

where $W'[\phi] = V[\phi]$ and coincides with the topological charge which is non-trivial for soliton states.

In order to find the Bogomol'nyi bound, Witten and Olive considered [4] the combinations

$$Q_+ + Q_- = \int \{ (\partial_+ \phi + V) \psi_- + (\partial_- \phi - V) \psi_+ \}, \quad (15)$$

$$Q_+ - Q_- = \int \{ -(\partial_+ \phi - V) \psi_- + (\partial_- \phi + V) \psi_+ \}. \quad (16)$$

Then, writing

$$\begin{aligned} 2M &= Z + (Q_+ + Q_-)^2, \\ 2M &= -Z + (Q_+ - Q_-)^2, \end{aligned} \quad (17)$$

one finds that the soliton mass M is bounded by the topological charge,

$$M \geq \frac{|Z|}{2}, \quad (18)$$

and that the bound is attained for those states $|\text{BPS}\rangle_\pm$ such that

$$(Q_+ + Q_-)|\text{BPS}\rangle_+ = 0 \quad (19)$$

or

$$(Q_+ - Q_-)|\text{BPS}\rangle_- = 0. \quad (20)$$

In view of the explicit form of charges these states correspond to kink solutions satisfying the first order BPS equations

$$\partial_0 \phi = 0, \quad \partial_1 \phi = V, \quad + \text{ kink}, \quad (21)$$

$$\partial_0 \phi = 0, \quad \partial_1 \phi = -V, \quad - \text{ anti-kink}, \quad (22)$$

which can be written in the form

$$\partial_+ \phi - V = 0, \quad \partial_- \phi + V = 0, \quad + \text{ kink}, \quad (23)$$

$$\partial_+ \phi + V = 0, \quad \partial_- \phi - V = 0, \quad - \text{ anti-kink}. \quad (24)$$

Each of the BPS kink solutions break half of the supersymmetry of the theory according to the choice among Eqs. (19) or (20).

Let us now study the supercurrent-supercurrent anticommutators. In particular, from the canonical commutation relations (11) one has

$$\{J_\alpha^\mu, \bar{Q}_\beta\} = 2i\gamma_{\alpha\beta}^\rho T_{\rho}{}^\mu + 2i\gamma_{5\alpha\beta} \xi^\mu \quad (25)$$

with J^μ the supercurrent and ξ^μ the topological current,

$$\xi^\mu = V \epsilon^{\mu\nu} \partial_\nu \phi, \quad (26)$$

related to the central charge through the identity

$$\int dx \xi^0 = Z. \quad (27)$$

Writing

$$\mathcal{M}_{\alpha\beta} = \{J_\alpha^1, \bar{Q}_\beta\} \quad (28)$$

one easily finds

$$\{\gamma^1, \mathcal{M}\} = 2\{\gamma^1, \gamma^v\} T_v^1 + 2i\{\gamma^1, \gamma^5\} \xi^1 \quad (29)$$

$$= -4T_{11}. \quad (30)$$

Explicitly, the l.h.s. takes the form

$$\{\gamma^1, \mathcal{M}\} = \begin{pmatrix} (J_{1-}, Q_-) - (J_{1+}, Q_+) & (J_{1+}, Q_-) - (J_{1-}, Q_+) \\ (J_{1+}, Q_-) - (J_{1-}, Q_+) & (J_{1-}, Q_-) - (J_{1+}, Q_+) \end{pmatrix} \quad (31)$$

and then

$$(J_{1-}, Q_-) - (J_{1+}, Q_+) = 4T_{11}, \quad (32)$$

$$(J_{1+}, Q_-) - (J_{1-}, Q_+) = 0. \quad (33)$$

From these two equations, one can write two identities for the stress-tensor

$$T_{11} = -\frac{1}{4}(J_{1+} + J_{1-}, Q_+ - Q_-), \quad (34)$$

$$T_{11} = -\frac{1}{4}(J_{1+} - J_{1-}, Q_+ + Q_-). \quad (35)$$

Then, in view of (49)–(50) and being the currents $J_{1\pm}$ given by

$$J_{1+} + J_{1-} = (\partial_+ \phi + V)\psi_- - (\partial_- \phi - V)\psi_+, \quad (36)$$

$$J_{1+} - J_{1-} = -(\partial_+ \phi - V)\psi_- - (\partial_- \phi + V)\psi_+, \quad (37)$$

we conclude that either

$$+\langle \text{BPS} | T_{11} | \text{BPS} \rangle_+ = 0 \quad (38)$$

or

$$-\langle \text{BPS} | T_{11} | \text{BPS} \rangle_- = 0. \quad (39)$$

That is, BPS saturated states preserving half of the supersymmetry correspond to states with vanishing stress tensor.

3. Scalar QED in three dimensions

Our conventions for γ -matrices, $(\gamma^\mu)_{\alpha\beta}$ are

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad (40)$$

$$\gamma^\mu \gamma^\nu = g^{\mu\nu} + i\epsilon^{\mu\nu\lambda} \gamma_\lambda,$$

with the metric with signature $(+ - -)$.

The $\mathcal{N} = 2$ supersymmetric action associated with the Abelian Higgs model is

$$\begin{aligned} S_{\mathcal{N}=2} = \int d^3x & \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu N) (\partial^\mu N) + \frac{1}{2} (D_\mu \phi)^* (D^\mu \phi) \right. \\ & - \frac{e^2}{4} N^2 |\phi|^2 - \frac{e^2}{8} (|\phi|^2 - \phi_0^2)^2 + \frac{i}{2} \bar{\Sigma} \not{\partial} \Sigma + \frac{i}{2} \bar{\psi} \not{\partial} \psi \\ & \left. - \frac{e}{2} N \bar{\psi} \psi - \frac{e}{2} (\bar{\psi} \Sigma \phi + \text{h.c.}) \right]. \end{aligned} \quad (41)$$

Concerning boson fields, A_μ is an Abelian gauge field, $F_{\mu\nu}$ its curvature, ϕ a complex scalar and N a real scalar field. The covariant derivative is defined as

$$D_\mu \phi = \partial_\mu \phi + ie A_\mu. \quad (42)$$

Note that the coupling constant in the gauge symmetry breaking scalar potential is taken as $\lambda = e^2/8$, the condition required in order to have $\mathcal{N} = 2$ supersymmetry. Fermion fields ψ and Σ are Dirac fermions and

$$\not{\partial} \psi = (i\not{\partial} - e\not{A}) \psi. \quad (43)$$

The energy-momentum tensor components of the bosonic sector are

$$\begin{aligned} T_{ij} = & \left(\frac{1}{2} B^2 - \frac{1}{2} |D_i \phi|^2 - \frac{e^2}{8} (|\phi|^2 - \phi_0^2)^2 \right) \delta_{ij} \\ & + \frac{1}{2} (D_i \phi)^* D_j \phi + \frac{1}{2} (D_j \phi)^* D_i \phi, \\ T_{00} = & \frac{1}{2} B^2 + \frac{1}{2} |D_i \phi|^2 + \frac{e^2}{8} (|\phi|^2 - \phi_0^2)^2, \end{aligned} \quad (44)$$

where $B = F_{12}$.

Action (41) is invariant under the following $\mathcal{N} = 2$ supersymmetry transformations

$$\begin{aligned} \delta A_\mu = & -i\bar{\eta}_c \gamma_\mu \lambda, \quad \delta \phi = \bar{\eta}_c \psi, \\ \delta \psi = & -i\gamma^\mu D_\mu \phi \eta_c - (8\lambda)^{1/2} N \phi \eta_c, \quad \delta N = \bar{\eta}_c \chi, \\ \delta \Sigma = & -\left(\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu} \gamma_\lambda + (2\lambda)^{1/2} (|\phi|^2 - \phi_0^2) + i\not{\partial} N \right) \eta_c, \end{aligned} \quad (45)$$

with η_c a complex (Dirac spinor) parameter.

The Noether supercurrent associated with invariance of action (41) under transformations (45) is

$$\begin{aligned} \mathcal{J}^\mu = & \bar{\eta}_c \left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu} \gamma_\lambda + i\not{\partial} N - \frac{e}{2} (|\phi|^2 - \phi_0^2) \right) \gamma^\mu \Sigma \\ & + \bar{\eta}_c \left(i(\not{\partial} \phi)^* - \frac{e}{2} N \phi^* \right) \gamma^\mu \psi + \bar{\psi} \gamma^\mu \left(-i\not{\partial} \phi - \frac{e}{2} N \phi \right) \eta_c \\ & + \bar{\Sigma} \gamma^\mu \left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu} \gamma_\lambda - i\not{\partial} N - \frac{e}{2} (|\phi|^2 - \phi_0^2) \right) \eta_c, \end{aligned} \quad (46)$$

so that the conserved charge \mathcal{Q} can be defined as

$$\mathcal{Q} = \frac{1}{\sqrt{2}e\phi_0} \int d^2x \mathcal{J}^0. \quad (47)$$

Writing

$$\mathcal{Q} = \bar{\eta}_c Q + \bar{\bar{Q}} \eta_c \quad (48)$$

one finds

$$\begin{aligned} Q = & \frac{1}{\sqrt{2}e\phi_0} \int d^2x \left[\left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu} \gamma_\lambda + i\not{\partial} N - \frac{e}{2} (|\phi|^2 - \phi_0^2) \right) \gamma^0 \Sigma \right. \\ & \left. + \left(i(\not{\partial} \phi)^* - \frac{e}{2} N \phi^* \right) \gamma^0 \psi \right] \end{aligned} \quad (49)$$

and

$$\begin{aligned} \bar{Q} = & \frac{1}{\sqrt{2}e\phi_0} \int d^2x \left[\bar{\Sigma} \gamma^0 \left(-\frac{1}{2} \epsilon^{\mu\nu\lambda} F_{\mu\nu} \gamma_\lambda - i\not{\partial} N - \frac{e}{2} (|\phi|^2 - \phi_0^2) \right) \right. \\ & \left. + \bar{\psi} \gamma^0 \left(-i\not{\partial} \phi - \frac{e}{2} N \phi \right) \right]. \end{aligned} \quad (50)$$

One can now compute the supersymmetry algebra among supercharges Q and \bar{Q} . Since this will be connected with the Bogomol'nyi bound for the Abelian Higgs model, we shall put $N = 0$ and, after using fermion anticommutator relations we shall also

put all fermions to zero. As one is interested in static configurations with finite energy one should also impose A_0 . The answer is

$$\{Q_\alpha, \bar{Q}^\beta\} = (\gamma_0)_\alpha{}^\beta P^0 + \delta_\alpha{}^\beta Z, \quad (51)$$

where

$$P^0 = \frac{1}{e^2 \phi_0^2} \int d^2x T_{00} \equiv M, \quad (52)$$

and the central charge Z is given by

$$Z = \frac{1}{2e^2 \phi_0^2} \int d^2x [eB(|\phi|^2 - \phi_0^2) + i\epsilon^{ij}(D_i\phi)(D_j\phi)^*]. \quad (53)$$

Here $i, j = 1, 2$.

One can see that the central charge (53) coincides with the topological charge (the quantized magnetic flux) of the vortex configuration. Indeed, Z can be rewritten in the form

$$Z = \int \partial_i \mathcal{V}^i d^2x, \quad (54)$$

where \mathcal{V}^i is given by

$$\mathcal{V}^i = \epsilon^{ij} \left(\frac{1}{2e} A_j + \frac{i}{2e^2 \phi_0^2} \phi^* D_j \phi \right), \quad (55)$$

so that, after using Stokes' theorem (and taking into account that $D_i\phi \rightarrow 0$ at infinity)

$$Z = \frac{1}{e} \oint A_i dx^i = \frac{\pi n}{e}, \quad (56)$$

with $n \in \mathbb{Z}$ an integer characterizing the homotopy class to which A_i belongs.

Let us now introduce the projector

$$\mathcal{P}_\pm = \frac{1}{2}(1 \mp \gamma_0) \quad (57)$$

and define

$$\mathcal{Q}_\pm = \mathcal{P}_\pm Q. \quad (58)$$

Then, we project Eq. (51) with \mathcal{P} and take the trace getting

$$\{\mathcal{Q}_{\pm\alpha}, \mathcal{Q}_{\pm\alpha}^\dagger\} = M \pm Z. \quad (59)$$

Taking the expectation value of (59) in an arbitrary state and since the anticommutator of an operator with its adjoint is a positive definite operator we conclude that

$$M \geq |Z| \quad (60)$$

or

$$M \geq \frac{\pi |n|}{e}, \quad (61)$$

which is the Bogomol'nyi bound for the vortex mass. For positive (negative) values of n the bound is attained only if the state is annihilated by \mathcal{Q}_+ (\mathcal{Q}_-),

$$\mathcal{Q}_\pm |BPS\rangle_\pm = 0. \quad (62)$$

In terms of components this is equivalent to the condition

$$(Q_+ \pm iQ_-) |BPS\rangle_\pm = 0. \quad (63)$$

In view of Eqs. (49)–(50), (57)–(58), Eq. (62) imply

$$B = \mp \frac{e}{2} (\phi_0^2 - |\phi|^2), \quad (64)$$

which are the BPS equations for the Abelian Higgs model. Due to (61), their solution also solves the static Euler–Lagrange equations

of motion. As in the kink case, according to the choice of sign in the BPS equations, the corresponding solution will break half of the supersymmetries. Let us finally insist that the condition $\lambda e^2 = \lambda/8$ necessary for this last fact, arises in the present approach from the requirement of $N = 2$ supersymmetry.

In order to connect supersymmetry with conditions on the stress tensor, we will analyze the supercurrent–supercharge algebra in the bosonic sector of the model. The relevant terms in the supercharge \bar{Q} and the spatial components \mathcal{J}_i of the supercurrent leading to (static) bosonic contributions are

$$\bar{Q} = \frac{1}{\sqrt{2}e\phi_0} \int d^2x \left\{ \bar{\Sigma} \left(-B - \frac{e}{2} \gamma^0 (|\phi|^2 - \phi_0^2) \right) - \bar{\psi} \epsilon^{ij} \gamma^i D_j \phi \right\} + \dots, \quad (65)$$

$$\begin{aligned} \mathcal{J}^i &= \left[iB\epsilon^{ij}\gamma^j - \frac{e}{2} (|\phi|^2 - \phi_0^2) \gamma^i \right] \Sigma^i \\ &\quad - [i(D_i\phi)^* - \epsilon^{ij}(D_j\phi)^*\gamma^0] \psi + \dots, \\ \bar{\mathcal{J}}^i &= \bar{\Sigma} \left[-iB\epsilon^{ij}\gamma^j - \frac{e}{2} (|\phi|^2 - \phi_0^2) \gamma^i \right] \\ &\quad + \bar{\psi} [i(D_i\phi) + \gamma^0 \epsilon^{ij}(D_j\phi)] + \dots, \end{aligned} \quad (66)$$

where we have written the supercurrent \mathcal{J}^i in the form

$$\mathcal{J}^i = \bar{\eta}_c \mathcal{J}^i + \bar{\mathcal{J}}^i \eta_c, \quad (67)$$

and ellipsis \dots indicate irrelevant terms which will be ignored from here on.

From Eqs. (65) and (66) we find that

$$\begin{aligned} \{J_\alpha^i, \bar{Q}_\beta\} &= \frac{\sqrt{2}}{e\phi_0} \left\{ \left(-\frac{1}{2} B^2 + \frac{e^2}{8} (|\phi|^2 - \phi_0^2)^2 + \frac{1}{2} |D_i\phi|^2 \right) \gamma_{\alpha\beta}^i \right. \\ &\quad \left. + \frac{1}{2} (\not{D}\phi)^* D_i\phi - (D_i\phi)^* \not{D}\phi \right\}_{\alpha\beta} \end{aligned} \quad (68)$$

and hence

$$\begin{aligned} \text{Tr}(\gamma^i \{J^j, \bar{Q}\}) &= \frac{2\sqrt{2}}{e\phi_0} \left\{ \left(B^2 - \frac{e^2}{2} (|\phi|^2 - \phi_0^2)^2 - |D_k\phi|^2 \right) \delta^{ij} \right. \\ &\quad \left. + (D^i\phi)^* D^j\phi + (D^j\phi)^* D^i\phi \right\}. \end{aligned} \quad (69)$$

Now, the r.h.s. is nothing but the symmetric stress tensor as defined in (44), so that

$$T_{ij} = \frac{e\phi_0}{2\sqrt{2}} \text{Tr}(\gamma_i \{J_j, \bar{Q}\}). \quad (70)$$

In particular we have

$$\{j_+^1 + ij_-^1, \bar{Q}_+ + i\bar{Q}_-\} = \frac{2\sqrt{2}}{e\phi_0} (T_{11} + iT_{21}), \quad (71)$$

$$\{j_+^1 - ij_-^1, \bar{Q}_+ - i\bar{Q}_-\} = -\frac{2\sqrt{2}}{e\phi_0} (T_{11} - iT_{21}), \quad (72)$$

$$\{j_+^2 + ij_-^2, \bar{Q}_+ + i\bar{Q}_-\} = \frac{2\sqrt{2}}{e\phi_0} (T_{12} - iT_{22}), \quad (73)$$

$$\{j_+^2 - ij_-^2, \bar{Q}_+ - i\bar{Q}_-\} = -\frac{2\sqrt{2}}{e\phi_0} (T_{12} + iT_{22}). \quad (74)$$

But

$$\begin{aligned} j_+^1 \pm ij_-^1 &= -\left(B \mp \frac{e}{2} (|\phi|^2 - \phi_0^2) \right) (\Sigma_+ \mp i\Sigma_-) \\ &\quad + (i(D_1\phi)^* \pm (D_2\phi)^*) (\psi_+ \pm i\psi_-), \end{aligned} \quad (75)$$

$$\begin{aligned} j_+^2 \pm ij_-^2 &= \pm i \left(B \mp \frac{e}{2} (|\phi|^2 - \phi_0^2) \right) (\Sigma_+ \mp i\Sigma_-) \\ &\quad + i(i(D_1\phi)^* \pm (D_2\phi)^*) (\psi_+ \pm i\psi_-). \end{aligned} \quad (76)$$

Then, analogously to the kink case, either one has $(\bar{Q}_+ + i\bar{Q}_-)|\text{BPS}\rangle_{\pm} = 0$, or $(j_+^1 + ij_-^1)|\text{BPS}\rangle_{\pm} = 0$ and $(j_+^2 + ij_-^2)|\text{BPS}\rangle_{\pm} = 0$ (a similar statement is valid for $(\bar{Q}_+ - i\bar{Q}_-)$, $(j_+^1 - ij_-^1)$, and $(j_+^2 - ij_-^2)$).

So we can write for BPS vortex states

$$\pm \langle \text{BPS} | T_{ij} | \text{BPS} \rangle_{\pm} = 0. \quad (77)$$

At this point, it should be stressed that Eq. (70) from which the vanishing of the stress tensor components was inferred is in general valid for other supersymmetric models in which one can write

$$T_{ij} = \mathcal{N}_d \text{Tr}(\gamma_i \{J_j, \bar{Q}\} + \gamma_j \{J_i, \bar{Q}\}), \quad (78)$$

where T_{ij} is the symmetric stress-tensor and \mathcal{N}_d a constant depending on the parameters of the specific model. This is valid for the kink (Eq. (25)), for the vortex (Eq. 70) but also for the dyon, the instanton taken as a soliton in $(4+1)$ -dimensions, etc. (see also [9–15]). In particular, consider the $3+1$ case, where the supercharge algebra for the $N=2$ Yang–Mills theory takes the form

$$\{Q^\alpha, \bar{Q}_\beta\} = -(\gamma_\mu)^\alpha_\beta P_\mu + (\gamma_5)^\alpha_\beta U + i\delta^\alpha_\beta V, \quad (79)$$

where $\alpha, \beta = 1, \dots, 4$ and the central charges U and V are surface integrals. If one takes as gauge group $O(3)$ and breaks this symmetry to $U(1)$ by giving a non-zero vacuum expectation value to the scalar field taken in the adjoint, U corresponds to the $U(1)$ magnetic charge and V to the electric charge. A Bogomol'nyi bound can be then derived from (79),

$$M^2 \geq U^2 + V^2 \quad (80)$$

and is saturated when the Bogomol'nyi–Prasad–Sommerfield equations are satisfied. Now, one can see that Eq. (78) holds in this case with the spatial components of the supercurrent taking the form

$$J_{ia} = \text{Tr}(\sigma^{\mu\nu} F_{\mu\nu} \gamma_i \Psi_a + \varepsilon_{ab} \not{D} \phi \gamma_i \Psi_b). \quad (81)$$

This formula corresponds to a bosonic sector containing a gauge field A_μ in the Lie algebra of $O(3)$ coupled to a Higgs scalar ϕ in the adjoint (there is an additional pseudoscalar field that should be put to zero to make contact with the Georgi–Glashow model). Concerning the fermion sector, Ψ_a ($a = 1, 2$) are two Majorana fermions. Then, using Eq. (78) and proceeding as for the kink and the vortex, one can see that Eq. (77) also holds for the Prasad–Sommerfield dyon. That is, the stress-tensor vanishes for BPS dyons, a fact that can be trivially confirmed by explicit computation of T_{ij} .

We have discussed in this note the relation between supersymmetry and the vanishing of the stress tensor for topological solitons in a variety of field theories in different space-time dimensions. Each one of the elements in this relation was already understood but our point was to show how they could be put together, by exploiting the relation that exists in supersymmetric theories between the supercurrent and the energy-momentum tensor. In fact, this relation was already underlying the analysis in Ref. [4] where BPS equations were derived from the relation between the supercharge algebra and the energy-momentum vector $P_\mu = \int d^3x T_{0\mu}$ which in the rest frame reduces to $P_0 = M$.

Here, we have instead used the fact that, since the supercurrent and the energy-momentum tensor belong to the same multiplet, we can extend the analysis of the relation between BPS states and

supersymmetry to the spatial components of J_μ and $T_{\mu\nu}$. If we consider for example the $d=3+1$ case in the superfield framework, the linear θ component of the multiplet is the supercurrent and the $\theta\theta$ component corresponds the energy-momentum tensor and they should then necessarily transform under supersymmetry one into the other,

$$[J_\mu, \bar{Q}] \propto \gamma^\nu T_{\mu\nu} + \dots \quad (82)$$

Similar identities hold in other $(d+1)$ -dimensional models. As signaled above, Eq. (78) leading to the connection between supersymmetric BPS states with the condition $T_{ij} = 0$ can be inferred from this formula. Now, as it is well known, T_{ij} gives the force f_i acting in a unit volume of the system. This, together with our result means that, in general, supersymmetry can guide the construction of non-interacting solitons bosonic models of interest just by considering the supersymmetric extension as a tool for identifying BPS states.

There are also possible applications of our observation in supergravity models, in connection with stability of cosmic strings [16, 17] and with the cosmological constant problem [18,19]. In particular, the so-called dominant energy condition, $T_{00} \geq |T_{ij}|$, valid for static spacetime, plays a central role to establish a connection between stability and the sign of the deficit angle [17]. In this context it is natural to study supergravity models with string-like BPS solutions in their bosonic sector. An analysis based on the supercharge algebra has been already presented [8] and it should be worthwhile to study the problem from the point of view of supercurrents presented here. We hope to report on these issues in a future work.

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